ESC194 Unit 6

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Abstract

Example:

$$e^x > 1 + x, x > 0$$

$$e^x = 1 + \int_0^x e^t dt > 1 + \int_0^x (1+t)dt = 1 + x + \frac{x^2}{2}$$

$$e^x = 1 + \int_0^x e^t dt > 1 + \int_0^x (1 + t + \frac{t^2}{2})dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ... + \frac{x^n}{n!}$$

$$\therefore e^x > 1 + x + \frac{x^2}{2}$$

1 6.4* General Logarithmic and Exponential Functions

$$x^{\frac{p}{q}} \to ln x^{\frac{p}{q}} = \frac{p}{q} \to \therefore e^{ln(x^{\frac{p}{q}})} = e^{\frac{p}{q} ln(x)} = x^{\frac{p}{q}}$$

Definition:

$$x^z = e^{zln(x)}, x > 0, z\text{z irrational}$$

Example:

$$3^{\sqrt{2}} = e^{\sqrt{2}ln3}$$

thus

$$x^{r+s} = x^r \cdot x^s; x^{r-s} = \frac{x^r}{x^s}; (x^r)^s = x^{r \cdot s}$$

$$\frac{d}{dx} x^p ppx^{p-1} \text{ for p rational}$$

$$\frac{d}{dx} x^r = rx^{r-1}$$

Proof

$$\frac{d}{dx} x^r = \frac{d}{dx} e^{rlnx} = e^{rlnx} \cdot \frac{r}{x} = x^r \frac{r}{x} = rx^{r-}$$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C$$

Example

$$\int \frac{xdx}{(x^2+1)^{\sqrt{2}}}$$

$$\text{Set: } n = x^2 + 1, du = 2xdx$$

$$\frac{1}{2} \int \frac{du}{u^{\sqrt{2}}} = \frac{1}{2}(\frac{1}{1-\sqrt{2}})(x^2+1)^{1-\sqrt{2}} + c$$

Example

$$f(x) = x^x$$

using:

$$x^x = e^{xlnx}$$

$$(x^x)' = e^{xlnx}(x \cdot \frac{1}{x} + lnx)$$

$$= x^x(1 + lnx)$$

Useful Strategy:

Given:

$$f(x) = p^x$$
$$(p^x)' = (e^{xlnp})' = ln(p)e^{elnp} = ln(p) \cdot p^x$$

$$\frac{d}{dx}p^n = p^n lnp \cdot \frac{du}{dx}$$

Example:

$$\frac{d}{dx}2^{3x^2} = 2^{3x^2} \cdot ln2 \cdot 6x$$

Example:

$$y = 2^{\frac{1}{x-1}} = e^{\frac{ln2}{x-1}} \rightarrow y' = e^{\frac{ln2}{x-1}} \cdot ln2 \cdot \frac{-1}{(x-1)^2}$$
$$= -2^{\frac{1}{x-1}} \cdot \frac{ln2}{(x-1)^2}$$

Identity?:

$$\int p^x dx = \frac{1}{lnp} \cdot p^x + c \text{ Given } p > 0, p \neq 1$$

Example

$$\int 3x^2 \cdot 7^{-5x^3}$$
$$\text{let } u = -5x^3$$
$$\therefore du = -15x^2$$
$$= \frac{-1}{5}\int 7^u du = -\frac{1}{5ln7} \cdot 7^u + c$$
$$= -\frac{1}{5ln7} \cdot 7^{-5x^3} + c$$

Log function to base p

$$f(g(x)) = \frac{ln(p^x)}{ln(p)} = \frac{xlnp}{lnp} = x$$

$$g(f(x)) = p^{\frac{lnx}{lnp}} = e^{lnp\frac{lnx}{lnp}} = e^{lnx} = x$$

Definition:

$$log_p(x) = \frac{lnx}{lnp}$$

Example:

$$log_{10}100 = \frac{ln100}{ln10} = \frac{ln10^2}{ln10} = \frac{2ln10}{ln10} = 2$$

Derivatives:

$$\frac{d}{dx}log_p(u(x)) = \frac{1}{ulnp}\frac{du}{dx}$$

$$\frac{d}{dx}log_p(x) = \frac{d}{dx}\frac{lnx}{lnp} = \frac{1}{lnp} \cdot \frac{1}{x}$$

$$f'(1) = \frac{1}{lnp}$$

Example:

$$\frac{d}{dx}log_7(2x^3 - x) = \frac{6x^2 - 1}{(2x^3 - x)} \cdot \frac{1}{ln7}$$

Estimating the value of e

Upper bound

$$ln(1 + \frac{1}{n}) = \int_1^{1+\frac{1}{n}} \frac{dt}{t} < \int_1^{1+\frac{1}{n}} 1dt$$

< since $\frac{1}{t} < \frac{1}{1}$ for all $t > 1$

$$ln(n + \frac{1}{n}) < \frac{1}{n} \rightarrow 1 + \frac{1}{n} < e^{\frac{1}{n}} \rightarrow (1 + \frac{1}{n})^n < e$$

Lower bound

$$ln(1 + \frac{1}{n}) = \int_1^{1+\frac{1}{n}\frac{dt}{t}} > \int_1^{1+\frac{1}{n}} \frac{dt}{1 + \frac{1}{n}} = \frac{1}{n+1}$$

> since $\frac{1}{t} > \frac{1}{1+\frac{1}{n}}$, i.e. $1 < t < 1 + \frac{1}{n}$

$$ln(1 + \frac{1}{n} > \frac{1}{n+1}), (1 + \frac{1}{n})^{n+1} > e$$

$$(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1}$$

Using different values of n and calculating lower and upper bound:

$$n = 1 \rightarrow 2, 4$$

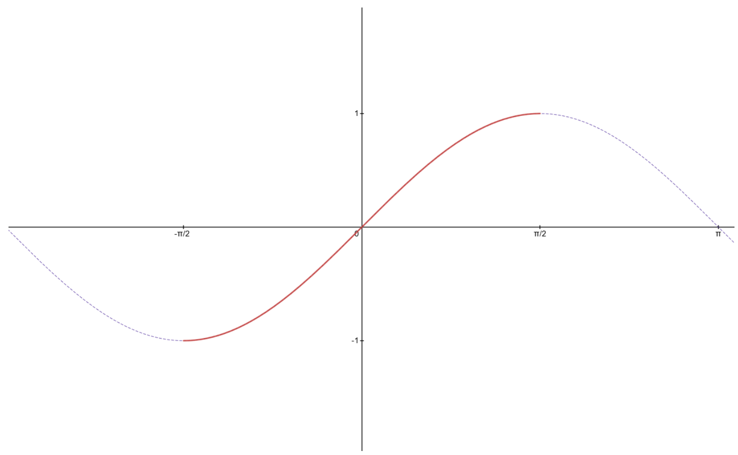
$$n = 2 \rightarrow 2.25, 3.375$$

$$n = 10 \rightarrow 2.59, 2.85$$

$$n = 10^6 \rightarrow 2.718280, 2.718283$$

$$e = \lim_{x \to \infty} (1 + \frac{1}{n})^n$$

2 6.6 Inverse Trigonometric Function



Totally fine to talk about inverse, horzintal line test passed:

$$sin^{-}1(x)$$

$$arcsin(x)$$

Domain: $[-1, 1]$ Range: $[-\frac{\pi}{2}], [32w]\frac{\pi}{2}]$